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A Generalized Multiplexer Theory

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Abstract—A general direct analytical design process is presented for multiplexers having any number of channels with arbitrary channel complexity, bandwidths, and interchannel spacings. The theory assumes initially that independent doubly terminated designs are available for the individual filters, and formulas for modifications to parameters associated with the first two resonators are developed to match the multiplexer. These formulas are approximate, and the limitations of the theory are indicated with several computed examples. The theory is applied to the design of a five-channel interdigital multiplexer.

A first-stage immittance compensation scheme is described which improves the design for limiting cases, but the theory of complete immittance compensation which handles even contiguous channel operation is reserved for a companion paper.

I. INTRODUCTION

IN TWO previous papers, direct design formulas were presented for bandpass channel diplexers [1], [2]. In this and a companion paper [3] the procedure is extended to the general multiplexer case having any number of channels, arbitrary channel complexity, and arbitrary channel bandwidths and center frequency allocations.

The theory may be developed in two distinct phases. In the first phase, to which this paper is devoted, design formulas are derived for interacting channel filters having direct connection (all in series or all in parallel) without additional immittance compensation networks. This is an important practical configuration and gives acceptable results for a wide variety of common specifications, as demonstrated by computer analysis and by experimental results presented. The main limitation is that the channels may not be spaced too closely in frequency.

In the second phase of the theory, consideration is given to the design of immittance compensation networks. Although a number of possible schemes for immittance

compensation are feasible, it has been found possible to design multiplexers on a manifold of uniform impedance where the phase shifts between the various filters on the manifold not only serve to separate the filters physically, but also act as immittance compensation networks. The results are expressed in the form of closed formulas, and little or no computer optimization is required. This extended theory may be applied even to the limiting case of contiguous band coverage and is the subject of the companion paper [3].

Initial consideration has been given to the possibility of designing multiplexers on the basis of exact synthesis, but it has become apparent that this is possible only for certain restrictive classes of networks. For example, in the diplexer case, if two networks have input impedances Z and $1 - Z$ and are connected in series to a resistive generator of 1-Ω internal impedance, then there is a perfect match at all frequencies at the input port. Power is distributed to the two networks as a function of frequency according to the frequency variation of Re Z and $1 -$ Re Z. If there is perfect transmission in one channel at the set of frequencies $\omega = \omega_i$, then there must be infinite attenuation in the other channel at $\omega = \omega_i$. Assuming that the set of ω_i are chosen such that there is equiripple transmission in the passbands, then, except for one very special case[1], the stopbands will not possess an equiripple behavior. In this example, there is no frequency region where both channels possess a common stopband. If they do, then the return loss at the common port will be finite except at a finite number of frequencies. This response may be made exactly equiripple in an optimum manner over the two individual passbands. However, the reflection at the individual channel outputs will not, in general, be equiripple. The only possible case in which this can be true is when

[1]This occurs when the minimum return loss level in the passband is approximately equal to the minimum insertion loss level in the stopband.

the opposite channel has infinite attenuation at the frequencies of perfect transmission. Thus, in general, optimum equiripple behavior simultaneously at the common port and the individual channel ports is not possible, and one may only approximate to this situation. For three channels and above, it may be argued that it is theoretically always impossible to have simultaneous equiripple reflection at the common port and individual channel ports simultaneously. The argument may be based upon the fact that all channels must provide attenuation poles at all frequencies of perfect transmission in every other channel leading to an incompatibility in the number of possible ripples in each passband. Thus only an approximate solution is possible. In practice this can be very good, since the out-of-band channels usually provide very high, if not ideally infinite, attenuation.

II. THEORY

As in the previous papers on diplexers [1], [2] the theory commences from lumped-element doubly terminated individual channel filters operating in isolation. Formulas are then developed which compensate for the interaction which takes place when the channels are connected.

There exists a large variety of lumped-element doubly terminated low-pass prototype filters, ranging from the conventional Chebyshev filter through to linear-phase filters with finite attenuation poles, e.g., [4]. These types of filters normally have an equiripple passband amplitude response with the maximum number of ripples. Thus there is perfect transmission at n points $\omega = \omega_i$, $i = 1 \to n$, where n is the degree of the transfer function. Defining the set of numbers ω_i and the level of the equiripple behavior uniquely defines the filter even for elliptic-function or linear-phase filters. Furthermore, the set of numbers ω_i represents a very sensitive description of the transfer function in the sense that small variations will cause significant changes in the transfer function response. Thus any modification to a filter due to interactions with any additional circuit elements must tend to preserve this set of frequencies ω_i at which perfect transmission occurs.

Optimum equiripple amplitude passband filters possess an electrically symmetric or antimetric realization. Additionally, a physically symmetrically prototype may always be synthesised using an even- and odd-mode decomposition [4]. For example, for the conventional Chebyshev low-pass prototype filter with an insertion loss response shown schematically in Fig. 1 and given by the formula

$$\frac{P_0}{P_L} = 1 + \epsilon^2 T_N^2(\omega'/\omega_1') \tag{1}$$

where

$$T_N(x) = \cos(N \cos^{-1} x)$$

we have the realization shown in Fig. 2, with the explicit design formulas [4],
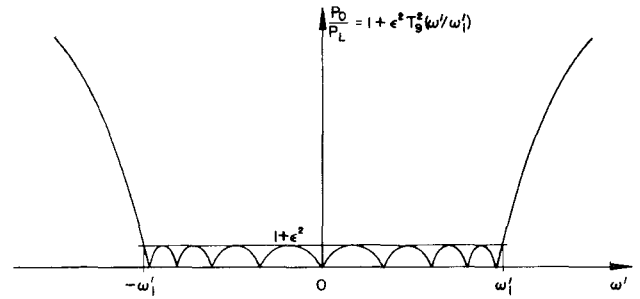

Fig. 1.   Insertion loss of Chebyshev low-pass prototype filter (positive and negative frequencies indicated).
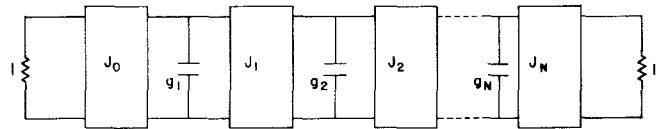

Fig. 2.   Realization of Chebyshev low-pass prototype filter using ideal admittance inverters and shunt capacitors.

$$g_k = \frac{2 \sin\left[\dfrac{(2k-1)\pi}{2N}\right]}{\gamma \omega_1'}$$

$$J_k = \frac{\sqrt{\gamma^2 + \sin^2\left(\dfrac{k\pi}{N}\right)}}{\gamma}$$

$$\gamma = \sinh\left\{\frac{1}{N}\sinh^{-1}\frac{1}{\epsilon}\right\}. \tag{2}$$

J_k is the characteristic admittance of the inverter between the kth and $(k+1)$th shunt capacitors. In a more general case, similar realizations exist, often with cross-coupling inverters between nonadjacent shunt capacitors. The theory which follows applies equally well to such filters, although slight modification will be necessary if a cross-coupling inverter at the first or second nodes is required. It may be noted that internal admittance scaling may be applied, and, for example, all of the main line inverters could possess unity characteristic admittance (i.e., $J_k = 1$). Occasionally it is more convenient not to do this, such as in the Chebyshev filter where the formulas given in (2) are used, and we shall always assume that $J_0 = 1$ for the unmodified (i.e., nonmultiplexed) filter.

The insertion loss characteristics of the n-channel multiplexer, indicating the insertion losses from the common port to the n output ports, is shown in Fig. 3. Here we have channel bandwidth W_r, centered at frequency Ω_r, $r = 1 \to n$. Note that the ripple levels of the channels are not required to be identical. The rth channel is derived from the low-pass prototype, such as the Chebyshev prototype of Fig. 1, using the frequency transformation

$$\omega' \to \frac{2\omega_1'}{W_r}(\omega - \Omega_r). \tag{3}$$

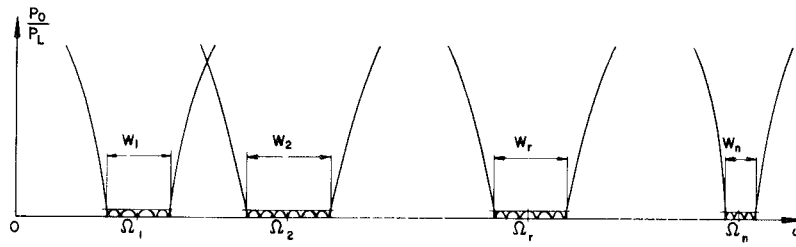This changes each prototype shunt capacitor g_k into a

Fig. 3. Insertion loss characteristics of the multiplexer, indicating notation for center frequencies and bandwidths.
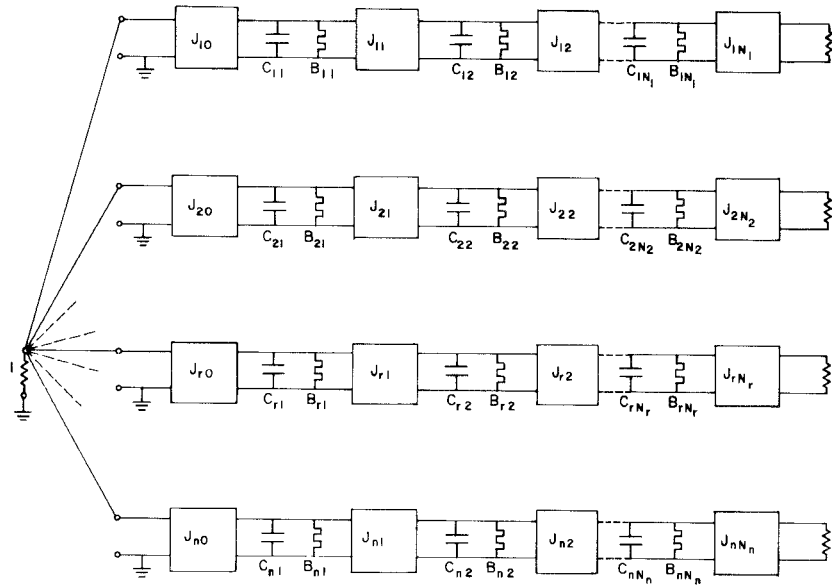

Fig. 4. Parallel-connected multiplexer.

capacitor

$$C_{rk} = \frac{2g_k \omega'_1}{W_r} \tag{4}$$

in parallel with a frequency invariant reactance

$$B_{rk} = \frac{-2\omega'_1 g_k \Omega_r}{W_r} = -C_{rk}\Omega_r. \tag{5}$$

Now consider scaling all of the center frequencies of each filter by a constant α, so that the frequency transformation (3) is modified to

$$\omega' \rightarrow \frac{2\omega'_1}{W_r}(\omega - \Omega_r \alpha). \tag{6}$$

This modification maintains the same bandwidth for each channel but changes the channel separations. Thus for α large, all of the guard bands are large, so that the interaction between filters in a parallel connection is reduced. In fact, as α tends to infinity the filters will not interact, and the individual passband performances will be maintained in the multiplexer. For each channel, an increase of 6 dB in the attenuation level over the passband regions of all other channels occurs, due to the potential-divider action resulting from the input admittance of the other channels being in parallel with that of the reference channel.

As α is decreased towards the design requirement of unity, the channels will interact. Any changes which are required in the element values of the channels may be expressed as a power series in α^{-1} with the leading terms being the original doubly terminated prototype values. The coefficients of the higher ordered terms may then be obtained such as to preserve the passband performances of each channel as α is decreased to unity. The criterion which is used is that the n frequencies ω_i of perfect transmission at both the common port and the appropriate channel port are preserved to a certain level of approximation. This criterion will preserve the individual channel performance since, as stated earlier, the behavior of any filter of the equiripple passband type is very sensitive with respect to the frequencies of perfect transmission.

In the simple case of parallel connection with no extra susceptance compensation networks, this process may be carried out exactly up to order α^{-2}, giving a residual error term of order α^{-3}, an expression for which may be derived.

The n-channel parallel-connected multiplexer is now shown in Fig. 4. The degree of the rth channel is denoted by N_r, and the kth shunt admittance of the rth channel is

given by

$$Y_{rk} = jC_{rk}(\omega - \Omega_r\alpha). \tag{7}$$

The next step is to carry out modifications to the element values in order to maintain the original frequencies of perfect transmission $(\Omega_r\alpha + \omega_i)$. Following the reasoning given in the previous papers [1], [2] where it is shown that the impedance inverters must be an even function of α, the frequency independent susceptances an odd function of α, and that the shunt capacitors may be retained unchanged, we have

$$J_{r0}^2 \to 1 - \gamma_{r02}\alpha^{-2} \tag{8}$$

$$J_{r1}^2 \to J_{r1}^2(1 - \gamma_{r12}\alpha^{-2}) \tag{9}$$

$$B_{r1} \to - C_{r1}(\Omega_r\alpha + \beta_{r11}\alpha^{-1} + \beta_{r13}\alpha^{-3}) \tag{10}$$

$$B_{r2} \to - C_{r2}(\Omega_r\alpha + \beta_{r23}\alpha^{-3}) \tag{11}$$

and all other elements are unchanged. The notation for the correction terms, γ_{rkl} or β_{rkl}, enables each such term to be associated with the rth channel, the kth element in that channel, and to a correction of order α^{-l}. If further correction terms such as β_{r21} or terms associated with a change in the values of the C_{rk} were to be included, then the following mathematics would show that such terms are redundant and may always be set to zero.

An expression for the input admittance to the common port of the multiplexer will now be derived. It is necessary to derive the in-band input admittance to the rth channel and the out-of-band admittances to the other $(n-1)$ channels within the frequency band of this rth channel. The required common port input admittance is then the sum of all these individual admittances.

The in-band input admittance to the rth channel in Fig. 4 at the original frequencies of perfect transmission $(\Omega_r\alpha + \omega_i)$ is given by

$$Y_r(\Omega_r\alpha + \omega_i) =$$

$$\frac{1 - \gamma_{r02}\alpha^{-2}}{jC_{r1}(\omega_i - \beta_{r11}\alpha^{-1} - \beta_{r13}\alpha^{-3}) + \dfrac{J_{r1}^2(1 - \gamma_{r12}\alpha^{-2})}{jC_{r2}(\omega_i - \beta_{r23}\alpha^{-3}) + G + jB}} \tag{12}$$

where $G + jB$ is the input admittance looking into the network remaining after the second shunt admittance. In deriving this equation we have included the modifications given by (8)–(11), and terms such as C_{rk} are still related to the original low-pass prototype element values by (4). The network represented by $G + jB$ is unmodified from the original, i.e., is independent of α.

The key element in the theory is now to express the fact that the input admittance is unity at the frequencies of perfect transmission for the original network, which is obtained by setting $\alpha = \infty$, i.e.,

$$1 = \frac{1}{jC_{r1}\omega_i + \dfrac{J_{r1}^2}{jC_{r2}\omega_i + G + jB}}$$

or

$$G + jB = \frac{J_{r1}^2}{1 - jC_{r1}\omega_i} - jC_{r2}\omega_i. \tag{13}$$

This may now be substituted into (12), giving

$$Y_r(\Omega_r\alpha + \omega_i) = \frac{1 - \gamma_{r02}\alpha^{-2}}{jC_{r1}(\omega_i - \beta_{r11}\alpha^{-1} - \beta_{r13}\alpha^{-3}) + \dfrac{J_{r1}^2(1 - \gamma_{r12}\alpha^{-2})}{\dfrac{J_{r1}^2}{1 - jC_{r1}\omega_i} - jC_{r2}\beta_{r23}\alpha^{-3}}}. \tag{14}$$

Expanding this expression as a power series in α^{-1} gives

$$Y_r(\Omega_r\alpha + \omega_i) = \frac{1 - \gamma_{r02}\alpha^{-2}}{jC_{r1}(\omega_i - \beta_{r11}\alpha^{-1} - \beta_{r13}\alpha^{-3}) + (1 - \gamma_{r12}\alpha^{-2})(1 - jC_{r1}\omega_i)\left[1 + \dfrac{jC_{r2}\beta_{r23}}{J_{r1}^2}(1 - jC_{r1}\omega_i)\alpha^{-3}\right]}$$

$$= \frac{1 - \gamma_{r02}\alpha^{-2}}{1 - jC_{r1}\beta_{r11}\alpha^{-1} - (1 - jC_{r1}\omega_i)\gamma_{r12}\alpha^{-2} + \alpha^{-3}\left[-jC_{r1}\beta_{r13} + \dfrac{jC_{r2}\beta_{r23}}{J_{r1}^2}(1 - jC_{r1}\omega_i)^2\right]}$$

$$= 1 + jC_{r1}\beta_{r11}\alpha^{-1} + \alpha^{-2}\left[(1 - jC_{r1}\omega_i)\gamma_{r12} - \gamma_{r02} - (C_{r1}\beta_{r11})^2\right]$$

$$+ \alpha^{-3}\left[jC_{r1}\beta_{r13} - j\frac{C_{r2}\beta_{r23}}{J_{r1}^2}(1 - jC_{r1}\omega_i)^2 + 2jC_{r1}\beta_{r11}(1 - jC_{r1}\omega_i)\gamma_{r12}\right]$$

$$- j\gamma_{r02}C_{r1}\beta_{r11} - j(C_{r1}\beta_{r11})^3\Big] + \epsilon(\alpha^{-4}) \tag{15}$$

where $\epsilon(\alpha^{-4})$ contains error terms in α^{-4} and below.

At the same set of frequencies $\Omega_r\alpha + \omega_i$ we may compute the input admittance of the mth out-of-band channel $(m \neq r)$ as

$$Y_m(\Omega_r\alpha + \omega_i) = \cfrac{1 - \gamma_{m02}\alpha^{-2}}{jC_{m1}\left[(\Omega_r - \Omega_m)\alpha + \omega_i - \beta_{m11}\alpha^{-1} - \beta_{m13}\alpha^{-3}\right] + \cfrac{J_{m1}^2(1 - \gamma_{m12}\alpha^{-2})}{jC_{m2}\left[(\Omega_r - \Omega_m)\alpha + \omega_i - \beta_{m23}\alpha^{-3}\right]}}. \tag{16}$$

Here the admittance includes only the first two shunt elements, since the remaining terms in this out-of-band situation become insignificant. Equation (16) may be expanded in a power series in α^{-1} as follows:

$$Y_m(\Omega_r\alpha + \omega_i) = \cfrac{1 - \gamma_{m02}\alpha^{-2}}{jC_{m1}(\Omega_r - \Omega_m)\alpha\left[1 + \cfrac{\omega_i\alpha^{-1} - \beta_{m11}\alpha^{-2}}{\Omega_r - \Omega_m} - \cfrac{J_{m1}^2\alpha^{-2}}{C_{m1}C_{m2}(\Omega_r - \Omega_m)^2}\right]}$$

$$= \cfrac{-j\alpha^{-1}}{C_{m1}(\Omega_r - \Omega_m)} \cdot \left\{1 - \cfrac{\omega_i\alpha^{-1}}{\Omega_r - \Omega_m} + \alpha^{-2}\left[-\gamma_{m02} + \cfrac{\beta_{m11}}{\Omega_r - \Omega_m} + \cfrac{J_{m1}^2}{C_{m1}C_{m2}(\Omega_r - \Omega_m)^2} + \cfrac{\omega_i^2}{(\Omega_r - \Omega_m)^2}\right]\right\} + \epsilon(\alpha^{-4})$$

$$\tag{17}$$

which is purely reactive apart from the error term. Summation of all these out-of-band admittances and adding to the in-band admittance (15) gives the final expression for the input admittance of the multiplexer at frequencies $(\Omega_r\alpha + \omega_i)$ as

$$Y_{\text{in}}(\Omega_r\alpha + \omega_i) = Y_r(\Omega_r\alpha + \omega_i) + \sum_{m=1 \neq r}^{n} Y_m(\Omega_r\alpha + \omega_i). \tag{18}$$
$$\hspace{2cm} (15) \hspace{3cm} (17)$$

By definition of the $\Omega_r\alpha + \omega_i$ as being the frequencies of perfect transmission, we may now set $Y_{\text{in}} = 1$. This gives a set of six equations for each channel consisting of a constant term in α^{-1}, a constant term and a ω_i term in α^{-2}, and a constant term, a ω_i term, and an ω_i^2 term in α^{-3}. Unfortunately there are only five parameters per channel (γ_{r02}, γ_{r12}, β_{r11}, β_{r13}, and β_{r23}) so that these equations cannot be satisfied exactly. As stated earlier, if any other parameters are introduced into the channel filters, then they are either redundant or create additional higher ordered terms in ω_i in the α^{-1}, α^{-2}, and α^{-3} coefficients. The solution to this dilemma is reserved for the second paper [3]. Here we will investigate the limitations on the technique given by a partial solution, which will leave residual terms in $\omega_i\alpha^{-3}$. The six sets of equations to be solved are as follows.

α^{-1} term:

$$C_{r1}\beta_{r11} - \sum_{m=1 \neq r}^{n} \frac{1}{C_{m1}(\Omega_r - \Omega_m)} = 0. \tag{19}$$

$\alpha^{-2}\omega_i$ term:

$$C_{r1}\gamma_{r12} - \sum_{m=1 \neq r}^{n} \frac{1}{C_{m1}(\Omega_r - \Omega_m)^2} = 0. \tag{20}$$

α^{-2} term:

$$\gamma_{r12} - \gamma_{r02} - (C_{r1}\beta_{r11})^2 = 0. \tag{21}$$

$\alpha^{-3}\omega_i^2$ term:

$$\frac{C_{r1}^2 C_{r2}\beta_{r23}}{J_{r1}^2} - \sum_{m=1 \neq r}^{n} \frac{1}{C_{m1}(\Omega_r - \Omega_m)^3} = 0. \tag{22}$$

α^{-3} term:

$$C_{r1}\beta_{r13} - \frac{C_{r2}\beta_{r23}}{J_{r1}^2}$$

$$+ 2C_{r1}\beta_{r11}\gamma_{r12} - \gamma_{r02}C_{r1}\beta_{r11} - (C_{r1}\beta_{r11})^3$$

$$- \sum_{m=1 \neq r}^{n} \left[\frac{-\gamma_{m02}}{C_{m1}(\Omega_r - \Omega_m)} + \frac{\beta_{m11}}{C_{m1}(\Omega_r - \Omega_m)^2}\right.$$

$$\left. + \frac{J_{m1}^2}{C_{m1}^2 C_{m2}(\Omega_r - \Omega_m)^3}\right] = 0. \tag{23}$$

Residual $\alpha^{-3}\omega_i$ term:

$$G_r = \left[-\frac{2C_{r1}C_{r2}\beta_{r23}}{J_{r1}^2} + 2C_{r1}^2\beta_{r11}\gamma_{r12}\right]\omega_i \tag{24}$$

where the input admittance is expressed as

$$Y_r(\Omega_r\alpha + \omega_i) = 1 + G_r\alpha^{-3} + \epsilon(\alpha^{-4}). \tag{25}$$

Equations (19)–(22) give the values of β_{r11}, γ_{r12}, γ_{r02}, and β_{r23} immediately for $r = 1 \to n$. These values may then be substituted into (23) to give a closed-form solution for the β_{r13} terms. Finally substitution into (24) gives the value of G_r as

$$G_r = 2\left[\sum_{m=1 \neq r}^{n} \frac{1}{C_{m1}(\Omega_r - \Omega_m)} \cdot \sum_{m=1 \neq r}^{n} \frac{1}{C_{m1}(\Omega_r - \Omega_m)^2}\right.$$

$$\left. - \frac{1}{C_{r1}} \sum_{m=1 \neq r}^{n} \frac{1}{C_{m1}(\Omega_r - \Omega_m)^3}\right]\omega_i. \tag{26}$$

In this case of a diplexer where $n = 2$, this expression

reduces to

$$G_1 = \frac{2}{C_{21}(\Omega_1 - \Omega_2)^3}\left(\frac{1}{C_{21}} - \frac{1}{C_{11}}\right)\omega_i \qquad (27)$$

with a similar equation for G_2. In the case of equal channels where $C_{21} = C_{11}$, then these terms vanish, as given also in [1, eq. (46)]. When the channels are unequal, addition of a susceptance compensation jB_0 at the input port where

$$B_0 = \frac{\alpha^{-1}}{\Omega_1 - \Omega_2}\left(\frac{1}{C_{21}} - \frac{1}{C_{11}}\right) \qquad (28)$$

gives complete compensation of G_1 and G_2, in agreement with [2, eq. (2)]. However, only the symmetrical diplexer may be matched to this extent without additional susceptance compensation networks. Initially it might be expected that the G_r terms of (24) and (25) with $\alpha = 1$ would give an estimate of the mismatch due to the approximations involved. Computer results have shown this not to be the case, and the $\epsilon(\alpha^{-4})$ terms in (25) tend to predominate.

III. PROTOTYPE RESULTS WITH AND WITHOUT ADMITTANCE COMPENSATION

The theory given in the previous section has been programmed, and several prototype multiplexers analyzed. From these results the following limitations have been determined.

1) The channels may not be spaced too closely in frequency. The relative spacing of channel B from channel A is defined as the ratio of the center frequency separation to the bandwidth of channel A. Relative spacings of 2:1 or more give excellent results, and quite acceptable results are obtained down to 1.5:1.

2) The channel return loss specification cannot be too high, since this results in values of C_{r1} ($r = 1 \rightarrow n$) which are relatively small, giving relatively large values for the correction terms of (19)-(23). As might be expected, the approximation is not as effective in this situation. In general, a return loss of approximately 20 dB gives acceptable results, but 26 dB or more is difficult to achieve.

3) There is an upper limit to the number of channels in the multiplexer, and the outer (i.e., lowest and highest) frequency channels deteriorate the most.

The examples shown in Figs. 5 and 6 serve to illustrate these limitations while also demonstrating the power and flexibility of the theory. Fig. 5 shows the common port return loss of multiplexers having three, five, seven, and nine channels. All channels consist of filters of degree 5 (i.e., five resonant circuits) with a return loss of 19.08 dB (VSWR 1.25:1). These multiplexers are symmetrical about zero frequency, i.e., the triplexer has channels centered at −3, 0, and 3; the quintuplexer channels are centered at −6, −3, 0, 3, 6, etc. The relative channel spacing is 1.5:1 in every case, i.e., a "borderline" situation. Only the positive frequencies are shown, since the negative frequency performance is identical. The nine-channel multiplexer is shown as the dotted-line graph

without the outer pair of channels, which are badly mismatched (to 2.3 dB).

The results show that when a pair of channels are added, the performance deteriorates, especially for these outer channels. On the other hand, the inner channels tend to improve or at worse are hardly affected. Thus the zero frequency channels are ideal in every case, and the ±3 frequency channels have a return loss of 15.5 dB in the triplexer but become ideal in the higher ordered multiplexers. The outer channels of the quintuplexer are mismatched to 10.4-dB return loss but improve to 17 dB for the higher ordered multiplexers. However, there is a limit to the effectiveness of this process of adding a pair of channels to match the inner ones, as seen by the fact that the channels at $f = \pm 9$ of the septaplexer are mismatched to 6.6 dB but improve only to 14 dB when two channels at $f = \pm 12$ are added. Note also that the channels at ±6 and ±9 have only four distinct return loss poles instead of five.

In spite of deficiencies in somewhat extreme cases, these results indicate that a great improvement in performance takes place when a pair of dummy channels are added to the multiplexer, giving a simple way of designing an admittance compensation network. Actually, it is not necessary to build complete filters for the admittance compensation "channels." Only the first two cavities need be added (in less severe cases only the first cavity) with the end cavity terminated in a short-circuit.

There is no significance to the fact that only odd-ordered multiplexers are shown in Fig. 5. Similar results are obtained with even-ordered multiplexers.

As stated earlier the theory is completely general in that highly asymmetric multiplexers may be designed with varying types of filters, bandwidths, and interchannel spacings within a single multiplexer. Fig. 6 shows a ten-channel multiplexer which illustrates these features to some extent. Actually, here the filters are identical in degree (= 5) and return loss (19.08 dB) but have different bandwidths and spacings. Cases where the filters are different also are given in [3]. The three numbers specifying each filter as shown in Fig. 6 are: channel number, center frequency, and bandwidth. It is seen that channels 3–7 are ideally matched, channels 8–10 are mismatched slightly but are acceptable in having all five return loss poles (and would probably tune better in practice), while only channels 1 and 2 are rather poorly matched. This is because these have quite close relative spacing for such a complex multiplexer. The outer channels at the high-frequency end have larger relative spacings.

It is worth noting that if doubly terminated filters are used without the compensation given in this paper, then multiplexers designed to the specifications of Figs. 5 and 6 would be badly mismatched, e.g., to 3-dB return loss.

We will now apply the prototype theory to the design of coaxial multiplexers. Application to the design of waveguide multiplexers is reserved for the companion paper [3].
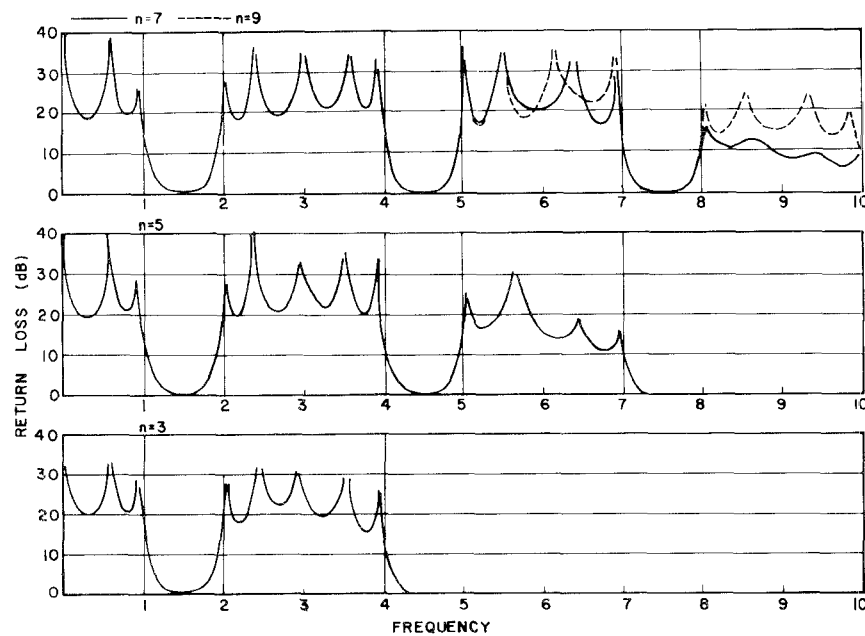
Fig. 5. Computed return loss of prototype multiplexers with three, five, seven, and nine channels (negative frequencies are not shown; the complete characteristics are symmetrical about zero frequency).
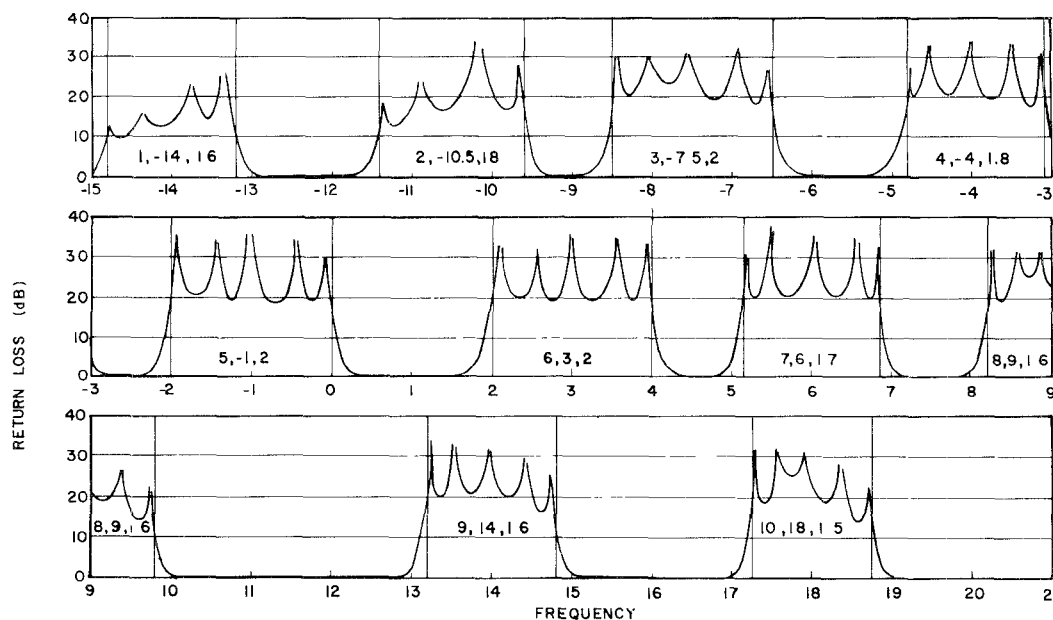

Fig. 6. Computed return loss of ten-channel multiplexer having nonuniform bandwidths and channel spacings. Channel parameters represent (channel number, center frequency, and bandwidth).

IV. DESIGN OF INTERDIGITAL MULTIPLEXERS ON A COMMON TRANSFORMER

Figs. 7(a) and (b) illustrate schematically combline and interdigital filters coupled by means of a common transformer. Common transformer diplexers of this type have been used for several years, although the scheme has been published only recently [5] (none of the diplexers described by Matthaei and Cristal [6] use a common transformer). It is possible to couple more than two filters to the common transformer by extending the planar struc-

tures shown in Fig. 7 into three dimensions, forming compact multiplexers.

There has been some confusion and controversy concerning the properties of the common transformer. Some designers regard the transformer as giving a series connection of the individual filters, others as a parallel connection. One object of this section is to present an exact equivalent circuit for the common transformer diplexer, at least in the case where the coupled bars on either side of the common junction are of equal electrical length. It is
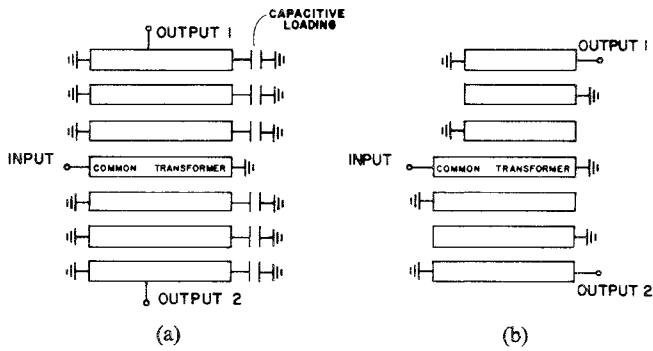
Fig. 7. Common-transformer diplexers, (a) combline, and (b) interdigital. The diagram shows coupled bars between ground planes (not shown).
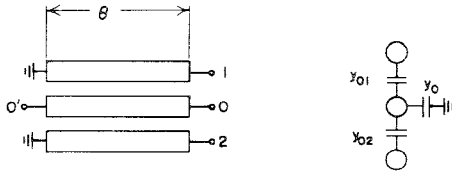

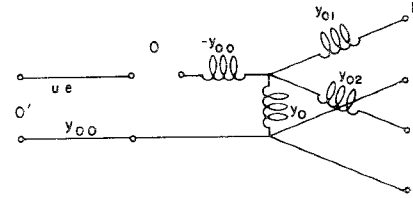Fig. 8. Basic equal coupled-bar circuit to be analyzed. θ is the electrical length of the bars.

Fig. 9. Equivalent circuit of Fig. 8, neglecting $1:-1$ transformers. Note that $y_{00}=y_0+y_{01}+y_{02}$. Inductors represent commensurate short-circuited stubs.
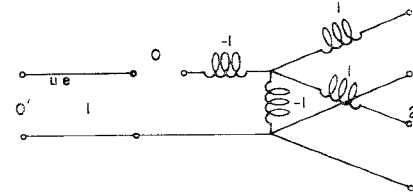

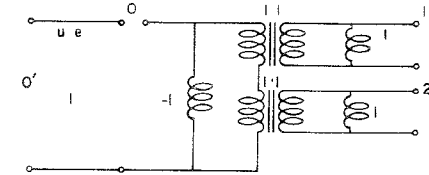Fig. 10. Circuit after application of the necessary restriction of (1).

Fig. 11. Pseudoseries equivalent circuit.

sufficient to formulate the equivalent circuit of the common transformer and the two adjacent coupled bars as shown in Fig. 8. Here the input port is labelled $0'$, and the output ports of the coupled bars are 1 and 2. The output port 0 of the common transformer is normally short-circuited to ground, but it is desirable to be able to locate this port in the equivalent circuit, since it may be used either to add on further filters or to add an immittance compensation network.

A. Series Representation

The equivalent circuit is found by writing down the six-port transfer matrix of the circuit and applying the short-circuit boundary conditions at ports $1'$ and $2'$. By terminating ports 1 and 2 in admittances Y_1 and Y_2, respectively, the two-port transfer matrix between ports $0'$ and 0 may then be derived. A unit element is now extracted using Richards' theorem, and the resulting equivalent circuit is shown in Fig. 9. Next, the following restrictive condition is applied, which is always a necessary condition for matching into the diplexer:

$$y_{00}=-y_{01}=-y_{02}=-1 \qquad (29)$$

and the circuit degenerates into that shown in Fig. 10. Using a star-delta transformation this circuit is converted to the more useful form of Fig. 11 which gives the nearest possible series representation of the circuit. This shows that port 0 is an exact series port in this circuit, but ports 1 and 2 are in series in only an approximate sense, being shunted by the short-circuited stub of negative characteristic admittance -1. This stub throws an admittance j cot θ across the line and may be regarded as having a small effect in some instances where Θ is near 90°, e.g., for two interdigital filters of small total bandwidth. However, in

the majority of cases, this stub cannot be neglected. Hence the common transformer connection is not a true series connection.

B. Parallel Representation

This is derived either from the six-port admittance matrix, or more directly, using the method of Sato and Cristal [7]. Applying the boundary conditions and using the simple equivalence of [7, Fig. 11(c)] the circuit of Fig. 12 results ($1:-1$ transformers at ports 1 and 2 are not shown). The immediate conclusion is that the common transformer network is not a good parallel connection when port 0 is open-circuited, but the equivalent circuit simplifies considerably when port 0 is short-circuited, giving the network shown in Fig. 13. It will be noted that this is the obvious parallel equivalent circuit formed from the transverse capacitance matrix of the three bars.

Applying the restriction of (29) to Fig. 13 it is easily shown that the series network of Fig. 11 with port 0 short-circuited is identical to the parallel network of Fig. 13. In fact, if impedances Z_1 and Z_2 are connected to ports 1 and 2, respectively, the input impedance at port $0'$ is

$$Z = \frac{Z_1 + Z_2 + 2Z_1 Z_2/t}{1 - Z_1 Z_2/t^2} \qquad (30)$$

where $t=j$ tan θ. Of course Z_1 and Z_2 are the driving point impedances of the individual filters of the diplexer.
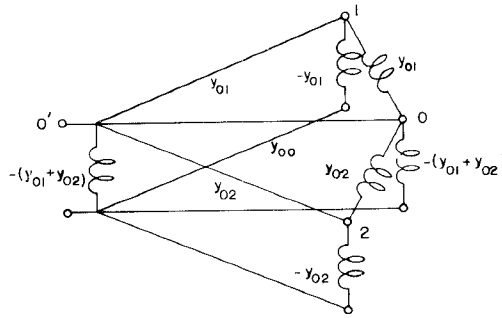
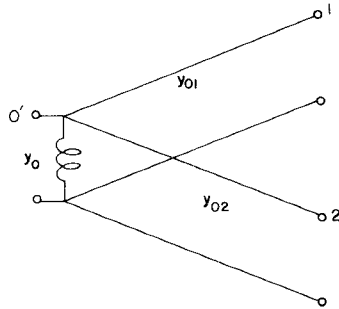Fig. 12. Pseudoparallel equivalent circuit.

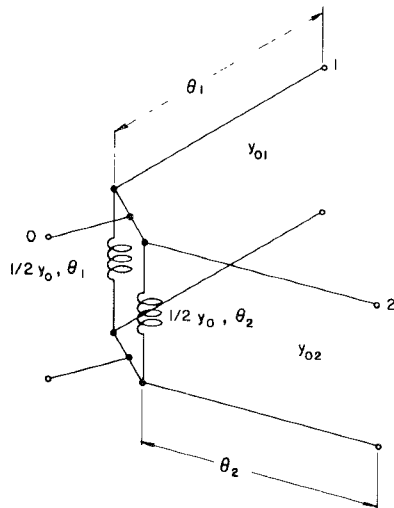Fig. 13. The parallel equivalent circuit with port 0 short circuited to ground.

Fig. 14. Approximate parallel equivalent circuit for the unequal bar case. The shunt stubs have unequal lengths as indicated.

The general case where bars 1 and 2 are of unequal length may be analyzed, e.g., by the Sato–Cristal method [7], but no simple equivalent circuit appears to result. It has been established that an extension of the simple parallel realization shown in Fig. 14 gives an excellent approximation for interdigital multiplexers of moderate bandwidth.

C. Example: A Five-Channel Interdigital Multiplexer

Fig. 15 shows the common port return loss of an X-band five-channel interdigital multiplexer, consisting of filters all of degree 8, return loss ripple of 19.08 dB, and

bandwidth of 400 MHz with interchannel spacings of 600 MHz. The multiplexer design is given later in this section, and the performance shown is for the actual distributed interdigital network. The solid-line graph shows the performance assuming that the input impedance of all filters were to add, i.e., it neglects the effect of the negative admittance stub shunted across the input, as shown in Fig. 11 for the diplexer case. The dotted line indicates the deterioration, mainly in the outer channels, caused by taking the more exact equivalent circuit into account. It is seen that this deterioration is not too severe and could well be tuned out in a practical situation. The performance is rather similar to the five-channel prototype multiplexer of Fig. 5 since the interchannel spacing is a uniform 1.5:1 in both cases, with the main difference being in the degree of the filters.

This multiplexer was built in the form of a star, as shown in the photograph of Fig. 16. As expected, the outer channels proved to be somewhat more difficult to match than the inner three channels, but finally the worst return loss obtained was 15.5 dB (VSWR 1.4:1). The interchannel isolation of this design is greater than 60 dB, and the insertion loss in the central 350 MHz of each passband was less than 1.2 dB, in good agreement with the theoretical prediction for silver-plated interdigital filters having 0.25-in ground plane spacing.

D. Design Equations for Interdigital Multiplexers

We will first review the equations for the interdigital filter having exact realization of the band edges, e.g., as given in [8]. A simplified derivation of these equations is to commence from the lumped-element prototype of Fig. 1 and to apply the lumped to distributed frequency variable transformation

$$\omega' = \frac{\tan \theta_1}{\tan \theta} \qquad (31)$$

with $\omega_1' = 1$ (for simplicity) where

$$\theta_1 = \frac{\pi}{2}\left(1 - \frac{w}{2}\right) \qquad (32)$$

and w is the fractional bandwidth of the filter, i.e., frequency bandwidth/center frequency. This gives the circuit of Fig. 17(a) where the inductors represent short-circuited shunt stubs of electrical length θ. Now an ideal admittance inverter is equivalent to a unit element bounded on each side by negative shunt stubs by virtue of the identity

$$\begin{bmatrix} 0 & j\sin\theta/Y \\ jY/\sin\theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ jY\cot\theta & 1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} \cos\theta & j\sin\theta/Y \\ jY\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ jY\cot\theta & 1 \end{bmatrix}. \qquad (33)$$

Hence, if each admittance inverter J_k is replaced by a unit element of admittance
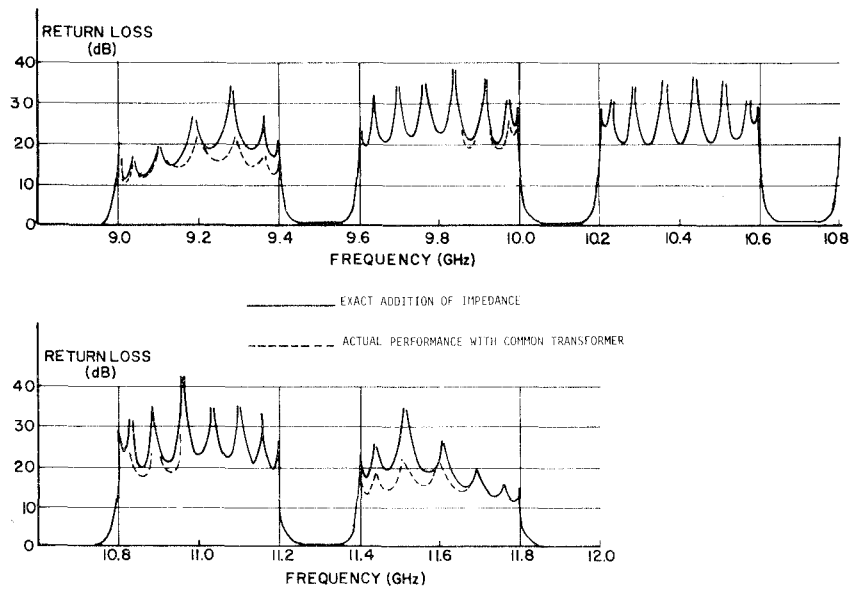
$$Y = J_k \sin \theta_1 \qquad (34)$$

Fig. 15. Theoretical characteristics of five-channel X-band interdigital multiplexer.
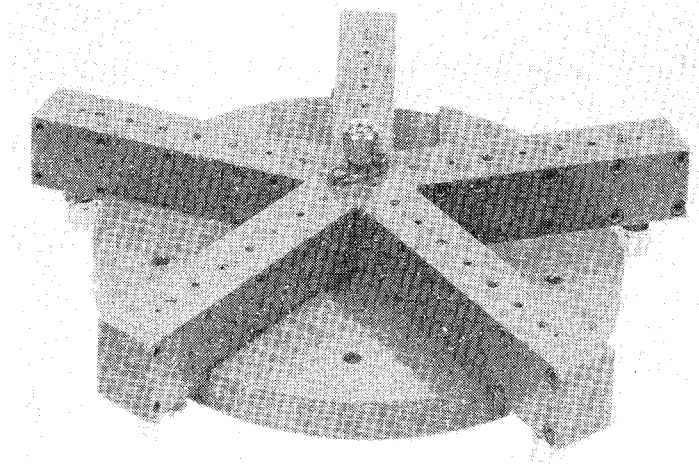

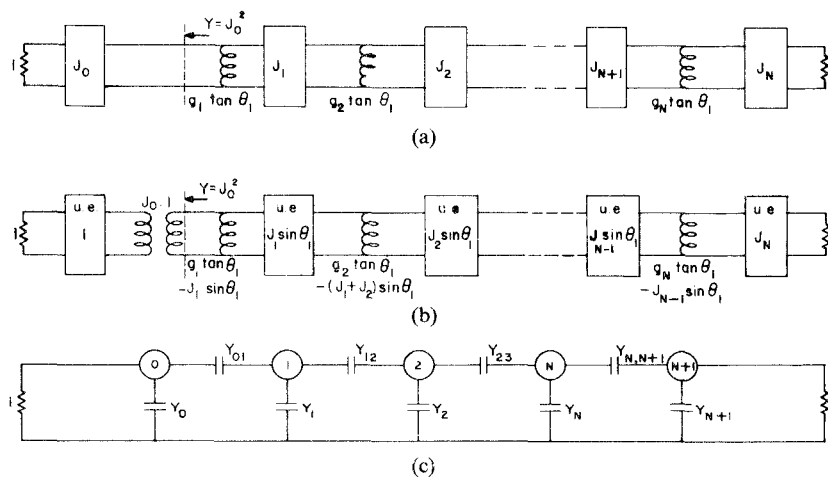Fig. 16. Five-channel X-band interdigital multiplexer.

Fig. 17. Derivation of design equations for an interdigital filter having exact realization of the band edges, (a) prototype filter, (b) approximate equivalent circuit, exact at $\theta = \theta_1$ and $\pi - \theta_1$, (c) cross section through interdigital filter indicating self- and mutual admittances.

bounded by shunt stubs of admittance $-Y$, then the ideal admittance inverter is realized exactly at the band edges (where $\theta = \theta_1$ and $\pi - \theta_1$), and the circuit of Fig. 17(b) results. The ideal admittance inverter J_0 is replaced by the equivalent circuit indicated. The nodal admittance matrix of the circuit to the right of the $J_0:1$ ideal transformer is

$$\frac{1}{J_0} \rightarrow \begin{bmatrix} g_1 \tan \theta_1 & -J_1 \sin \theta_1 & 0 & \\ -J_1 \sin \theta_1 & g_2 \tan \theta_1 & -J_2 \sin \theta_1 & \\ & & & \ddots \\ 0 & -J_2 \sin \theta_1 & g_3 \tan \theta_1 & \\ & & \ddots & & \ddots \end{bmatrix} .$$

$$\downarrow 1/J_0$$

(35)

The ideal transformer is removed by multiplying the first row and column by $1/J_0$, as indicated in (35), and adding the redundant unit element at the input results in the admittance matrix

$$Y_{k,k+1} = \frac{J_k \cos \theta_1}{\sqrt{g_k g_{k+1}}}, \qquad k = 1 \cdots N - 1 \tag{39}$$

$$Y_{N,N+1} = \frac{J_N}{\sqrt{g_N \tan \theta_1}} \tag{40}$$

$$Y_0 = 1 - Y_{01} \tag{41}$$

$$Y = 1 + \frac{J_0^2}{g_1 \tan \theta_1} - Y_{01} - Y_{12} \tag{42}$$

$$Y_k = 1 - Y_{k-1,k} - Y_{k,k+1}, \qquad k = 2 \cdots N - 1 \tag{43}$$

$$Y_N = 1 + \frac{J_N^2}{g_N \tan \theta_1} - Y_{N-1,N} - Y_{N,N+1} \tag{44}$$

$$Y_{N+1} = 1 - Y_{N,N+1}. \tag{45}$$

These values may be multiplied by $7.534/\sqrt{\epsilon_r}$ (ϵ_r being the relative dielectric constant of the medium) to obtain normalized self- and mutual capacitances C/ϵ, as given in the many well-known design graphs and equations in the literature. Here ϵ is the dielectric constant of the medium.

$$\downarrow \frac{J_0}{\sqrt{g_1 \tan \theta_1}} \qquad \downarrow \frac{1}{\sqrt{g_2 \tan \theta_1}}$$

$$\begin{array}{c} \frac{J_0}{\sqrt{g_1 \tan \theta_1}} \rightarrow \\ \\ \frac{1}{\sqrt{g_2 \tan \theta_1}} \rightarrow \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 + \frac{g_1}{J_0^2} \tan \theta_1 & -\frac{J_1}{J_0} \sin \theta_1 & 0 \\ 0 & -\frac{J_1}{J_0} \sin \theta_1 & g_2 \tan \theta_1 & -J_2 \sin \theta_1 \\ & & \ddots & & \ddots \\ 0 & 0 & -J_2 \sin \theta_1 & 0 \\ & & & \ddots \end{bmatrix} . \tag{36}$$

Multiplying rows and columns as indicated in (36) gives a physically realizable admittance matrix, namely,

$$\begin{bmatrix} 1 & \frac{-J_0}{\sqrt{g_1 \tan \theta_1}} & 0 & 0 \\ \frac{-J_0}{\sqrt{g_1 \tan \theta_1}} & 1 + \frac{J_0}{g_1 \tan \theta_1} & -\frac{J_1 \cos \theta_1}{\sqrt{g_1 g_2}} & 0 \\ 0 & -\frac{J_1 \cos \theta_1}{\sqrt{g_1 g_2}} & 1 & -\frac{J_2 \cos \theta_1}{\sqrt{g_2 g_3}} \\ & \ddots & & \ddots & & \ddots \end{bmatrix} \tag{37}$$

leading immediately to the design equations for the self- and mutual admittances of the interdigital network of Fig. 17(c), i.e.,

$$Y_{01} = \frac{J_0}{\sqrt{g_1 \tan \theta_1}} \tag{38}$$

These equations may now be extended to the interdigital multiplexer with a common transformer as shown schematically in Fig. 18, which indicates the central transformer region of a quadruplexer. The initial step is to replace J_0 and J_1 in (38) and (39) by the modified values
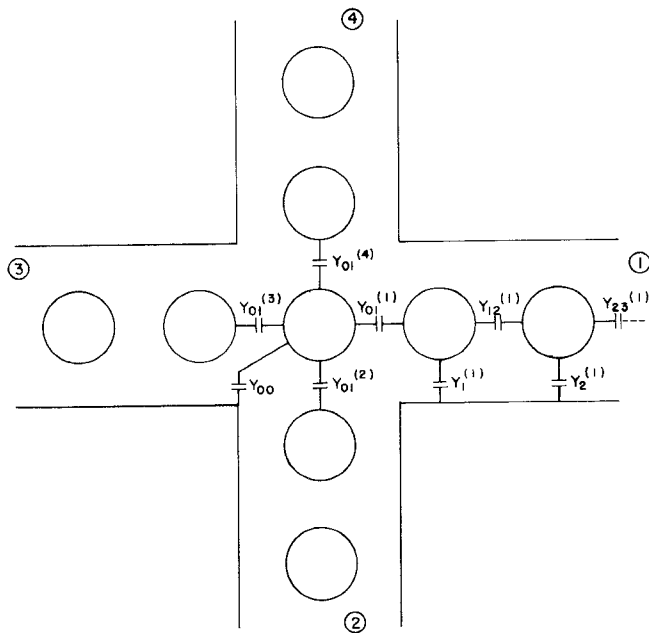
Fig. 18. Cross section through central region of an interdigital quadruplexer.

given by (8) and (9), while (41) is replaced by

$$Y_{00} = 1 - \sum_{r=1}^{n} Y_{01}^{(r)} \tag{46}$$

where $Y_{01}^{(r)}$ is the value (38) for the rth filter. This gives complete information for design of the self- and mutual capacitances. Finally, the first two resonators of each filter will be detuned according to (10) and (11). For example, the first resonator of each filter would in isolation have electrical length $\pi/2$ at midband, but for the rth filter within the multiplexer this changed to the value

$$\theta_{r1} = \frac{\pi/2}{1 + \beta_{r11} + \beta_{r13}}. \tag{47}$$

Similarly, the midband electrical lengths of the second set of resonators become

$$\theta_{r2} = \frac{\pi/2}{1 + \beta_{r23}} \tag{48}$$

where $r = 1, 2, \cdots, n$.

It will be observed from inspection of (46) that the multiplexer may not always be realizable in this common transformer format, since Y_{00} may become negative. The fact that Y_{00} is small theoretically actually assists in realizing a structure such as shown in cross section by Fig. 18, since the transformer is almost completely shielded from ground by the surrounding elements, and Y_{00} will be quite small.

As an illustration of the design, the self- and mutual capacitances of the five-channel multiplexer previously described in this section (Figs. 15 and 16) are given in Table I. The values are conventionally normalized by

TABLE I
SELF- AND MUTUAL CAPACITANCES OF FIVE-CHANNEL
MULTIPLEXER (SELF-CAPACITANCE OF COMMON TRANSFORMER
IS 0.998)

Self and Mutual Capacitances	CHANNEL CENTER FREQUENCIES				
	9 2	9.8	10 4	11 0	11 6
C_1	6.080	6 337	6 399	6.404	6 225
C_2	7.197	7.218	7 236	7 253	7.267
C_3	7 241	7 259	7 275	7 289	7.302
C_4	7 253	7 270	7 285	7 299	7 311
C_5	7 253	7.270	7 285	7 299	7 311
C_6	7 241	7 259	7 275	7 289	7 302
C_7	7.175	7 197	7.217	7 234	7 250
C_8	6 218	6 252	6 289	6 323	6.354
C_9	6 181	6 223	6 261	6 296	6.329
C_{01}	1 606	1 215	1 139	1.147	1 430
C_{12}	0 186	0 174	0 164	0 155	0 147
C_{23}	0 151	0 142	0 134	0.126	0 120
C_{34}	0 142	0 133	0 125	0 118	0 112
C_{45}	0 140	0 131	0 124	0 117	0.111
C_{56}	0 142	0 133	0 125	0 118	0 112
C_{67}	0 151	0 142	0 134	0 126	0 120
C_{78}	0 208	0.195	0 184	0 174	0 165
C_{89}	1 353	1 311	1 273	1 230	1 205

multiplying the admittances of (38)–(46) by 7.534, as stated earlier.

Further theoretical and practical examples (particularly for waveguide manifold-type multiplexers) are deferred to the companion paper [3].

V. CONCLUSIONS

Direct design formulas have been derived for completely general noncontiguous multiplexers without addition of immittance compensation networks. In this process only parameters associated with the first two resonators of each channel filter are changed compared with double-terminated filters acting in isolation (8)–(11), and the five sets of correction terms are given by simple closed formulas (19)–(23). The theory is confirmed both by computer analysis of several multiplexers and by a practical five-channel multiplexer. Limitations of the design process are discussed, and immittance compensation by means of dummy channels was described.

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Design of General Manifold Multiplexers

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Abstract—The direct analytical design process for arbitrary multiplexers given in a previous paper is extended to the case of bandpass channel filters connected to a uniform-impedance manifold (e.g., a length of waveguide or transmission line). The previous approximations are greatly improved by adding immittance compensation in a way which not only preserves the canonic form of the network but also assists in the physical construction by spacing the filters along a manifold. The phase shifters between channels are themselves sufficient to compensate the filter interactions to such an extent that contiguous channeling cases are designable. The results are presented mainly in closed form requiring minimal computer optimization.

Analysis of multiplexers with frequency-dependent manifolds indicate that there are restrictions on the total bandwidth, but a ten-channel multiplexer is probably feasible, suitable for input and output multiplexers required in typical communications systems. Practical results on a simple manifold triplexer are presented.

I. INTRODUCTION

IN THE previous paper [1] it was shown that there are inherent limitations to the canonic matching of a multiplexer consisting of a number of filters connected directly in series or parallel. We may define canonic matching as that requiring only changes to the parameters of the filters and not adding extra immittance compensation networks. In this paper immittance compensation is introduced, but in a way which not only preserves the canonic form of the network but also assists in the physical construction by spacing the filters along a manifold.

The phase shifters between channels are sufficient to compensate the multiplexer to such an extent that contiguous channeling cases are designable by the theory.

It is interesting to consider various approaches to the design of multiplexers, particularly on a manifold feed. Due to requirements in communication satellites and elsewhere, many attempts have been made to produce such multiplexers. One important and difficult requirement is that of an output multiplexer on a waveguide manifold with bandpass channels separated to yield guardbands of only 10 percent. Most design techniques have adopted an approach based upon singly terminated bandpass channels resulting in 3-dB crossover points between channels, e.g., [2], [3]. Such designs exhibit good return loss over the channel bandwidths and the guardbands. Also, dummy channels have to be included to simulate channels at the edges of the total multiplexer bandwidth, forming an additional annulling network. Thus redundant elements are necessary, and the channel interactions are compensated to produce a channel performance comparable to the individual channels based upon a singly terminated prototype.

The need for contiguous band multiplexers originally arose in receiver design for countermeasures where the incoming signal was unknown and complete band coverage was necessary with good match at all frequencies. Here all channels have to be designed on a singly terminated basis and must provide a prescribed level of attenuation over the major part of other bands.

However, the requirements for multiplexers in communication systems are different since they must provide